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LETTER TO THE EDITOR

Periodic reduction of the factorization chain and the Hahn polynomials

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Abstract. The $N = 4$ periodic closure of the factorization chain is considered. It is shown that the nonlinear operator algebra corresponding to this closure can be transformed into the quadratic Hahn algebra. As a result, the three-term recurrence coefficients for the Hahn polynomials provide a special realization of the $N = 4$ periodic factorization chain.

The factorization method was developed by Schrödinger as a convenient tool for solving the eigenvalue problem

$$L\psi(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x)\right)\psi(x) = \lambda\psi(x) \tag{1}$$

for the particular potentials $u(x)$. Within this method, one replaces (1) by the chain of problems

$$L_j\psi_j(x) = \lambda\psi_j(x) \qquad j = 0, \pm 1, \pm 2, \ldots \tag{2}$$

whose Hamiltonians L_j satisfy the intertwining relations

$$L_j A_j^+ = A_j^+ L_{j+1} \qquad A_j^- L_j = L_{j+1} A_j^- \tag{3}$$

where $A_j^{\pm} = \pm\mathrm{d}/\mathrm{d}x + f_j(x)$. Resolution of the constraints (3) leads to the representation

$$L_j = A_j^+ A_j^- + \lambda_j \tag{4}$$

where λ_j are some constants, and to the abstract factorization chain [1]

$$A_{j+1}^+ A_{j+1}^- + \lambda_{j+1} = A_j^- A_j^+ + \lambda_j. \tag{5}$$

Equivalently, one may start from factorization (4) and use (5) for the definition of the Hamiltonian L_{j+1} which automatically guarantees relations (3). In general, the operators A_j^+ and A_j^- need not be Hermitian conjugates of each other. A useful fact is that for the

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special solutions of the factorization chain (5), the constants λ_j provide discrete spectra of the Hamiltonians L_j.

Among the various reductions of the chain (5), there is a very simple reduction fixed by the following periodicity condition:

$$L_{j+N} = L_j + \mu \qquad \lambda_{j+N} = \lambda_j + \mu \tag{6}$$

where μ is some constant. It has been investigated in detail in [2] for the case when $A_j^{\pm}$ are the first-order differential operators and $L_j = -\mathrm{d}^2/\mathrm{d}x^2 + u_j(x)$. A more general periodic closure was considered in [3], namely

$$L_{j+N} = U^{+} L_j U + \mu \qquad \lambda_{j+N} = \lambda_j + \mu \tag{7}$$

where U is a unitary shift operator. If one defines the operators $L \equiv L_1$,

$$B^{+} = A_1^{+} A_2^{+} \dots A_N^{+} U^{+} \qquad B^{-} = U A_N^{-} A_{N-1}^{-} \dots A_1^{-} \tag{8}$$

the conditions (7) then imply that

$$\begin{aligned} &[L, B^{\pm}] = \pm\mu B^{\pm} \\ &B^{+}B^{-} = \prod_{k=1}^{N}(L - \lambda_k) \\ &B^{-}B^{+} = \prod_{k=1}^{N}(L + \mu - \lambda_k). \end{aligned} \tag{9}$$

The operators L, $B^{\pm}$ therefore form a polynomial algebra: for $N = 1$ it is the oscillator algebra and for $N = 2$ it coincides with $su(1, 1)$.

We consider here the case when $N = 4$ and show that the corresponding symmetry algebra can be transformed into the so-called Hahn algebra $QH(3)$ considered in [4, 5].

For this, let us introduce the operators

$$\begin{aligned} &K_1 = L \\ &K_2 = B^{+} + B^{-} + \alpha L^2 + \beta L + \gamma \\ &K_3 = \mu(B^{+} - B^{-}) \end{aligned} \tag{10}$$

where α, β and γ are some constants to be determined. From (10) and (9), we obtain

$$[K_1, K_2] = K_3 \tag{11a}$$

$$[K_3, K_1] = \mu^2(\alpha K_1^2 + \beta K_1 + \gamma - K_2) \tag{11b}$$

$$\begin{aligned} [K_2, K_3] = {} & 2\mu[B^{-}, B^{+}] - 2\alpha^2\mu^2 K_1^3 - 3\alpha\beta\mu^2 K_1^2 + \alpha\mu^2\{K_1, K_2\} + \beta\mu^2 K_2 \\ & - (2\gamma\alpha + \beta^2)\mu^2 K_1 - \beta\gamma\mu^2 \end{aligned} \tag{11c}$$

where $\{K_1, K_2\} = K_1K_2 + K_2K_1$. The commutator $[B^{-}, B^{+}]$ can be found from (9)

$$[B^{-}, B^{+}] = \mu(4K_1^3 + (6\mu - 3s_1)K_1^2 + (4\mu^2 - 3\mu s_1 + 2s_2)K_1 + \mu^3 - \mu^2 s_1 + \mu s_2 - s_3) \tag{12}$$

where

$$s_1 = \sum_{k=1}^{4} \lambda_k \qquad s_2 = \sum_{k<\ell} \lambda_k \lambda_\ell \qquad s_3 = \sum_{j<k<\ell} \lambda_j \lambda_k \lambda_\ell \qquad s_4 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \tag{13}$$

are the standard symmetric functions.

From (11*c*) and (12), we see that if

$$\alpha = \pm 2 \qquad \beta = \pm 2\mu \mp s_1 \tag{14}$$

the terms $\sim K_1^3$ and $\sim K_1^2$ on the RHS of (11*c*) disappear and we obtain

$$\begin{aligned} [K_1, K_2] &= K_3 \\ [K_3, K_1] &= AK_1^2 + BK_1 + C_2 K_2 + D_2 \\ [K_2, K_3] &= A\{K_1, K_2\} + BK_2 + C_1 K_1 + D_1 \end{aligned} \tag{15}$$

where the structure constants A, B, C_i and D_i are

$$A = \pm 2\mu^2 \qquad B = \pm \mu^2 (2\mu - s_1) \qquad C_2 = -\mu^2 \qquad D_2 = \mu^2 \gamma$$

$$C_1 = \mu^2 (4\mu^2 - 2\mu s_1 - s_1^2 + 4 s_2 \mp 4\gamma) \qquad D_1 = 2\mu^2 (\mu^3 - \mu^2 s_1 + \mu s_2 - s_3 \mp \gamma(\mu - s_1/2)).$$

The algebra with the commutation relations (15) is called the Hahn algebra $QH(3)$ [4, 5]. The general symmetry algebra (9), for $N = 4$, can thus be reduced to the Hahn (quadratic) algebra $QH(3)$.

There are many interesting consequences that follow from this observation. Let us list some of them.

(i) If all zero modes of the operator B^- satisfy the necessary boundary conditions, then the discrete spectrum of the operator K_1 contains four arithmetic series (we assume that each A_j^- has only one zero mode). This is a simple corollary of (9).

(ii) A less obvious consequence of (15) is that the spectrum of K_2 may be either finite and quadratic in the number of level, or may contain both discrete and continuous parts.

(iii) The overlaps between eigenstates of the operators K_1 and K_2 are expressed in a special case in terms of the ordinary finite-dimensional Hahn polynomials.

One can thus expect that the Hahn polynomials provide a special realization of the $N = 4$ closure condition (7). Investigation of corollaries of facts (i)–(iii) for the realization of algebra (9) with the help of the standard Painlevé equations [2] lies beyond the scope of the present paper. Let us consider instead the following realization of the operators $A_j^{\pm}$:

$$A_j^- |n\rangle = Q_n(j)|n-1\rangle + |n\rangle \qquad A_j^+ |n\rangle = R_n(j)|n\rangle + |n+1\rangle \tag{16}$$

where $|n\rangle$ is some abstract basis of states. The operators L_j then become finite-difference Schrödinger operators

$$L_j |n\rangle = |n+1\rangle + u_n(j)|n-1\rangle + b_n(j)|n\rangle \tag{17}$$

where

$$u_n(j) = Q_n(j) R_{n-1}(j) \qquad b_n(j) = Q_n(j) + R_n(j) + \lambda_j. \tag{18}$$

The refactorization condition (5) leads to the following *discrete dressing chain* [6, 7]:

$$
\begin{aligned}
&Q_n(j+1)R_{n-1}(j+1) = Q_n(j)R_n(j)\\
&Q_n(j+1) + R_n(j+1) = Q_{n+1}(j) + R_n(j) + \lambda_j - \lambda_{j+1}.
\end{aligned} \tag{19}
$$

As shown in [7], equations (19) define a special form of the discrete-time Toda lattice.

We impose the following closure condition [6]:

$$
R_n(j+4) = R_{n+2}(j) \qquad Q_n(j+4) = Q_{n+2}(j) \qquad \lambda_{j+4} - \lambda_j = \mu \tag{20}
$$

which corresponds to (7) with the operator U acting as the shift operator

$$
U|n\rangle = |n+2\rangle. \tag{21}
$$

In general, the analysis of conditions (19) and (20) leads to complicated nonlinear difference equations which can be considered as difference analogues of the Painlevé equations (see [2, 6]). We propose here a direct method for showing how the recurrence coefficients of the Hahn polynomials arise as special solutions of these conditions.

The finite-dimensional Hahn polynomials of the discrete argument x

$$
P_n(x;\alpha,\beta,N) = (-1)^n\, {}_3F_2\left[\begin{matrix} -n, -x, n+\alpha+\beta+1 \\ \alpha+1, -N+1 \end{matrix}\middle| 1\right] \tag{22}
$$

obey the three-term recurrence relation [8]

$$
d_n P_{n-1} + g_n P_{n+1} + (g_n + d_n) P_n = x P_n \tag{23}
$$

with

$$
g_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-1-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \tag{24a}
$$

$$
d_n = \frac{n(n+\beta)(n+\alpha+\beta+N)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \tag{24b}
$$

where

$$
0 \leqslant x, n \leqslant N-1. \tag{25}
$$

A renormalization of the polynomials allows one to rewrite relation (23) in the form

$$
v_{n+1}\psi_{n+1} + \psi_{n-1} + c_n\psi_n = x\psi_n \tag{26}
$$

$$
\begin{aligned}
&v_n(\alpha,\beta,N) = d_n g_{n-1}\\
&c_n(\alpha,\beta,N) = d_n + g_n.
\end{aligned} \tag{27}
$$

Consider the eigenvalue problem for the operators L_j (17)

$$
L_j|\Phi\rangle_j = x|\Phi\rangle_j \qquad |\Phi\rangle_j = \sum_n \psi_n(j)|n\rangle. \tag{28}
$$

The coefficients $\psi_n(j)$ are found as solutions of the discrete Schrödinger equations

$$u_{n+1}(j)\psi_{n+1}(j) + \psi_{n-1}(j) + b_n(j)\psi_n(j) = x\psi_n(j).$$

The relations (5) now determine a chain of discrete Darboux transformations for the wavefunctions $\psi(j)$. From the action of operators A_j^-, $A_j^-|\Phi)_j \propto |\Phi)_{j+1}$, one finds

$$\psi_n(j+1) = \psi_n(j) + Q_{n+1}(j)\psi_{n+1}(j) \tag{29}$$

$$u_n(j+1) = u_{n+1}(j)\frac{Q_n(j)}{Q_{n+1}(j)} \tag{30a}$$

$$b_n(j+1) = b_n(j) + Q_{n+1}(j) - Q_n(j) \tag{30b}$$

where new eigenfunctions $\psi_n(j+1)$ obey a discrete Schrödinger equation with recurrence coefficients $u_n(j+1)$ and $b_n(j+1)$ and the same eigenvalue x. Note that the wavefunction's mapping given in (29) is defined up to a normalization factor; we are here only interested in the transformation of recurrence coefficients. In this context, the closure condition (20) means that after four successive Darboux transformations, the discrete potentials $u_n(j)$ and $b_n(j)$ should be recovered up to the shift $n \to n+2$ and the addition of μ to b_n.

In the case of the Hahn polynomials, one can find at least four simple solutions to equations (19) that are analytical in j. We shall call these factorization wisps. They are presented below with the normalization $u_n(j=0) \equiv v_n(\alpha,\beta,N)$, $b_n(j=0) \equiv c_n(\alpha,\beta,N)$.

(i) The first wisp is

$$Q_n(j) = \frac{n(n+\beta)(N-n)}{(2n+\alpha_j+\beta)(2n+\alpha_j+\beta+1)}$$

$$R_n(j) = \frac{(n+\alpha_j+\beta+1)(n+\alpha_j+1)(n+\alpha_j+\beta+N+1)}{(2n+\alpha_j+\beta+1)(2n+\alpha_j+\beta+2)}$$

$$u_n(j) = v_n(\alpha_j,\beta,N) \qquad b_n(j) = c_n(\alpha_j,\beta,N)$$

where

$$\alpha_j = \alpha + j \qquad \lambda_j = -\alpha - j - 1. \tag{31}$$

This solution corresponds to the following transformation of the recurrence coefficients for the Hahn polynomials:

$$v_n(\alpha,\beta,N) \to v_n(\alpha+1,\beta,N)$$

$$c_n(\alpha,\beta,N) \to c_n(\alpha+1,\beta,N).$$

(ii) The second wisp is

$$\begin{aligned} Q_n(j) &= -\frac{n(n+\alpha+\beta_j+N_j)(n+\alpha)}{(2n+\alpha+\beta_j)(2n+\alpha+\beta_j+1)} \\ R_n(j) &= -\frac{(n+\beta_j+1)(n+\alpha+\beta_j+1)(N_j-n-1)}{(2n+\alpha+\beta_j+1)(2n+\alpha+\beta_j+2)} \\ u_n(j) &= v_n(\alpha,\beta_j,N_j) \qquad b_n(j) = c_n(\alpha,\beta_j,N_j) \end{aligned} \tag{32}$$

where

$$\beta_j = \beta + j \qquad N_j = N - j \qquad \lambda_j = N - j - 1. \tag{33}$$

One step of this transformation corresponds to the following change of the recurrence coefficients:

$$v_n(\alpha, \beta, N) \to v_n(\alpha, \beta + 1, N - 1)$$
$$c_n(\alpha, \beta, N) \to c_n(\alpha, \beta + 1, N - 1).$$

(iii) The third wisp is

$$\begin{aligned} Q_n(j) &= -\frac{(n + j + \alpha + \beta_j)(n + j + \beta_j)(n + j + \alpha + \beta_j + N)}{(2n + 2j + \alpha + \beta_j)(2n + 2j + \alpha + \beta_j + 1)} \\ R_n(j) &= -\frac{(n + j + 1)(n + j + \alpha + 1)(N - 1 - n - j)}{(2n + 2j + \alpha + \beta_j + 1)(2n + 2j + \alpha + \beta_j + 2)} \\ u_n(j) &= v_{n+j}(\alpha, \beta_j, N) \qquad b_n(j) = c_{n+j}(\alpha, \beta_j, N) \\ \beta_j &= \beta - j \qquad \lambda_j = \beta + N - j - 1 \end{aligned} \tag{34}$$

and the corresponding transformations are

$$v_n(\alpha, \beta, N) \to v_{n+1}(\alpha, \beta - 1, N)$$
$$c_n(\alpha, \beta, N) \to c_{n+1}(\alpha, \beta - 1, N).$$

(iv) The fourth wisp is

$$\begin{aligned} Q_n(j) &= \frac{(n + j + \alpha_j + \beta)(n + j + \alpha_j)(N_j - n - j)}{(2n + 2j + \alpha_j + \beta)(2n + 2j + \alpha_j + \beta + 1)} \\ R_n(j) &= \frac{(n + j + 1)(n + j + \beta + 1)(n + j + \alpha_j + \beta + N_j + 1)}{(2n + 2j + \alpha_j + \beta + 1)(2n + 2j + \alpha_j + \beta + 2)} \\ u_n(j) &= v_{n+j}(\alpha_j, \beta, N_j) \qquad b_n(j) = c_{n+j}(\alpha_j, \beta, N_j) - j \\ \alpha_j &= \alpha - j \qquad N_j = N + j \qquad \lambda_j = -j - 1. \end{aligned} \tag{35}$$

Note that only in this case does the Darboux transformation lead to an additive factor for c_n.

None of the above wisps satisfy the closure condition (20). However, there is a lot of freedom in the definition of the discrete time evolution of the recurrence coefficients. One can consider an evolution which mixes up the four wisps in an arbitrary manner. Consider, for example, the following sequence of Darboux transformations:

$$\text{(i)} \to \text{(ii)} \to \text{(iii)} \to \text{(iv)}. \tag{36}$$

It is easy to see that it leads to the following transformation of the recurrence coefficients of the Hahn polynomials:

$$\begin{aligned} v_n(\alpha, \beta, N) &\to v_{n+2}(\alpha, \beta, N) \\ c_n(\alpha, \beta, N) &\to c_{n+2}(\alpha, \beta, N) - 1 \end{aligned} \tag{37}$$

and that we find, in fact, the $N = 4$ periodic closure conditions (20) with $\mu = -1$. The periodicity conditions (20) are thus satisfied when the discrete time j has a different meaning than in wisps (i)–(iv), enumerating rather the steps in (36). Evidently, there are other combinations of (i)–(iv) that lead to periodic closure, for instance, the cyclic permutation of the steps in (36) gives the same result.

To conclude, we have proved that the Hahn polynomials provide a specific realization of the factorization chain with the $N = 4$ periodic closure condition (20). The symmetry algebra (9) is in this case equivalent to the quadratic Hahn algebra $QH(3)$. This gives one more example of systems associated with classical orthogonal polynomials that belong to the infinite family of Hamiltonians with formal discrete spectra composed from an arbitrary number of arithmetic or geometric series appearing after the (q-)periodic reduction of the discrete dressing chain [6].

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